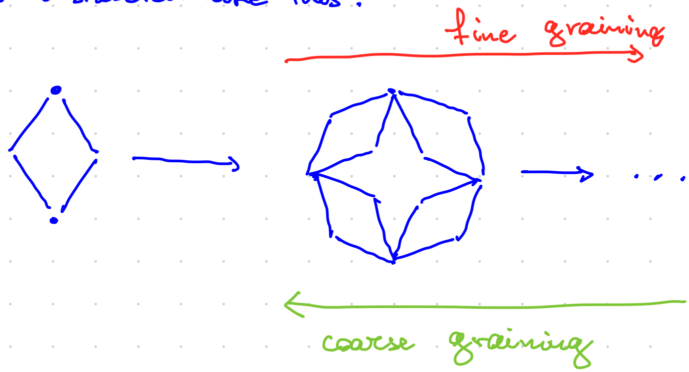
We are now going to work a few examples.

Diamond (or "hierarchical") lattice

The diamond lattice is an artificial construct that allows an exact implementation of the RG because it shares with the 1D lattice the absence of the proliferation of the interactions.
It is constructed like this:



For simplicity, we consider only the case with $h=0$
We also recall the coarse graining of the 1D Ising model:

$$e^{4K^{(n+1)}} = \left[\cosh 2K^{(n)}\right]^2$$

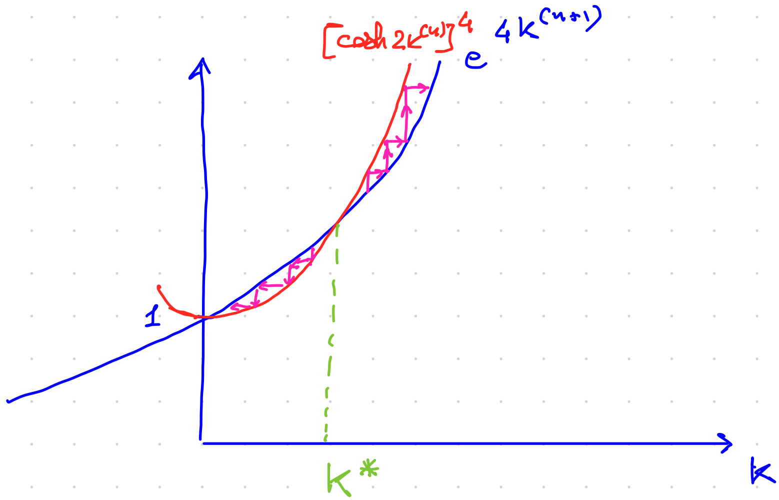
and we observe that the coarse graining operation

corresponds to two, independent, 1D coarsegrainings.

This in turn corresponds to the following recursion, iterative, equation

$$e^{4k^{(n+1)}} = [\cosh 2k^{(n)}]^4$$

Let's look graphically at the solution:



$[\cosh 2k \;]^4$ has zero derivative at the origin (hence, it stays below e^{4k} locally), but it grows like e^{8k} asymptotically, hence it goes above e^{4k}, and the two must cross in k^*

Graphically it is also clear that K^* is unstable under iteration, while by converse $K=0$ $(T=\infty)$ and $K=\infty$ $(T=0)$ are stable fixed points.

K^* thus corresponds to the critical point!

Let's compute the associated eigenvalue

$$e^{4K^{(n+1)}} = \left[\cosh(2K^{(n)})\right]^4$$

$$\Downarrow$$

$$K^{(n+1)} = \ln\cosh(2K^{(n)})$$

and

$$K^{(n+1)} = K^* + \delta K^{(n+1)} \qquad K^{(n)} = K^* + \delta K^{(n)}$$

Then

$$\cancel{K^*} + \delta K^{(n+1)} = \cancel{\ln\cosh(2K^*)} + 2\tanh(2K^*)\,\delta K^{(n)} + O(\delta K^2)$$

and we have

$$\delta K^{(n+1)} = 2\tanh(2K^*)\,\delta K^{(n)}$$

is

$$2\tanh(2K^*) > 1 \ ?$$

In order to prove it, let's take the derivative of the iterative equation

$$e^{k^{(n+1)}} = \cosh(2k^{(n)})$$

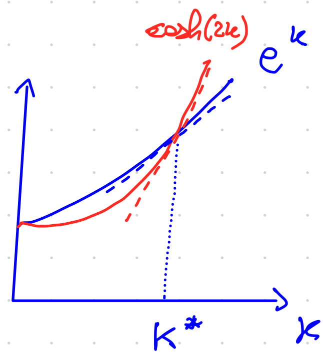
$$\downarrow$$

$$e^{k^*}(\cancel{1+}\,\delta k^{(n+1)}) = \cancel{\cosh(2k^*)} + 2\sinh(2k^*)\,\delta k^{(n)}$$

$$\downarrow$$

$$e^{k^*}\,\delta k^{(n+1)} = 2\sinh(2k^*)\,\delta k^{(n)}$$

But graphically:



The slope (tangent) of $\cosh(2k)$ is larger than the one of e^k in k^*

This means that $2\,\dfrac{\sinh(2k^*)}{e^{k^*}} > 1$

But $e^{k^*} = \cosh(2k^*)$ $\Longrightarrow$ $\boxed{2\,\mathrm{th}(2k^*) > 1}$

The rescaling factor is $b = 2$, just as for the 1D case, which means

$$b^{y_t} = 2\tanh(2K^*) \qquad \text{with } b = 2$$

$$\Downarrow$$

$$y_t = \log_2[2\tanh(2K^*)]$$

Renormalization of the ϕ^4 model

We recall the Hamiltonian of the ϕ^4 model:

$$H(\Phi(\vec{x})) = \int d\vec{x}\left[c(\vec{\nabla}\phi)^2 + r\phi^2 + u\phi^4\right]$$

In this expression, β is included in the interactions, as we have already seen.

Then we must write the partition function we then must coarse grain:

$$Z = \int \mathcal{D}\phi \, e^{-H}$$

But this expression is not computable, and it is even not clear how to perform the coarse graining.

Let's simplify the problem in 2 steps

1) Let's for one moment forget about the quartic terms

$$H = \int d\vec{x} \left[c(\vec{\nabla}\phi)^2 + r\phi^2 \right]$$

and

$$Z = \int \mathcal{D}\phi \; e^{-\int d\vec{x} \left[c(\vec{\nabla}\phi)^2 + r\phi^2 \right]}$$

We have already worked on this previously, but we want now to focus on a renormalisation group approach.

How do we implement the coarse-graining?

2) Fourier space

$$\tilde{\phi}(\vec{k}) = \int d\vec{x}\, \phi(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}$$

$$\phi(\vec{x}) = \frac{1}{(2\pi)^d} \int_0^{\Lambda} d\vec{k}\, \tilde{\phi}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

$\Lambda = \frac{1}{a}$ is the smallest scale in the system.

Then

$$(\vec{\nabla}\phi)^2 = \left[\int \frac{d\vec{k}}{(2\pi)^d} (-i\vec{k})\, e^{i\vec{k}\vec{x}}\, \tilde{\phi}(\vec{k}) \right]^2 =$$

$$= \int \frac{d\vec{k}\, d\vec{k}'}{(2\pi)^{2d}} (-\vec{k}\cdot\vec{k}')\, e^{i(\vec{k}+\vec{k}')\vec{x}}\, \tilde{\phi}(\vec{k})\tilde{\phi}(\vec{k}')$$

and

$$\int d\vec{x}\, (\vec{\nabla}\phi)^2 = \int \frac{d\vec{k}\, d\vec{k}'}{(2\pi)^{2d}} (-\vec{k}\cdot\vec{k}')\, \tilde{\phi}(\vec{k})\, \tilde{\phi}(\vec{k}') \underbrace{\int d\vec{x}\, e^{i(\vec{k}+\vec{k}')\vec{x}}}_{2\pi\,\delta(\vec{k}+\vec{k}')} =$$

$$= \frac{1}{(2\pi)^d} \int_0^{\Lambda} d\vec{k}\; \vec{k}^2\, \tilde{\phi}(\vec{k})\, \tilde{\phi}(-\vec{k}) =$$

$$= \frac{1}{(2\pi)^d} \int_0^{\Lambda} d\vec{k}\; \vec{k}^2\, \tilde{\phi}(\vec{k})\, \tilde{\phi}^*(\vec{k})$$

The quadratic part works similarly

$$\int d\vec{x}\; \phi^2(\vec{x}) = \frac{1}{(2\pi)^d}\int_0^\Lambda d\vec{k}\; \tilde{\phi}(\vec{k})\,\tilde{\phi}^*(\vec{k})$$

Together we have

$$H = \frac{1}{(2\pi)^d}\int_0^\Lambda d\vec{k}\,(ck^2 + r)\,\tilde{\phi}(\vec{k})\,\tilde{\phi}^*(\vec{k})$$

Let us now write the partition function

$$Z = \int \mathcal{D}\tilde{\phi}\; e^{-\frac{1}{(2\pi)^d}\int_0^\Lambda d\vec{k}\,(ck^2+r)\,\tilde{\phi}(\vec{k})\,\tilde{\phi}^*(\vec{k})}$$

Coarse-graining the system means "tracing out" the small scales: we integrate between $\frac{\Lambda}{b}$ and Λ with $b > 1$ (but not much larger).

We can divide the Hamiltonian in two parts

$$H = H_< + H_>$$

with

$$H_< = \frac{1}{(2\pi)^d}\int_0^{\Lambda/b} d\vec{k}\; (ck^2+r)\,\tilde{\phi}(\vec{k})\,\tilde{\phi}^*(\vec{k})$$

$$H_> = \frac{1}{(2\pi)^d}\int_{\Lambda/b}^{\Lambda} d\vec{k}\; (ck^2+r)\,\tilde{\phi}(\vec{k})\,\tilde{\phi}^*(\vec{k})$$

and the partition function becomes

$$Z = \int \mathcal{D}\phi_< \, \mathcal{D}\phi_> \, e^{-H_< - H_>} =$$

$$= \underbrace{\int \mathcal{D}\phi_< \, e^{-H_<} \int \mathcal{D}\phi_> \, e^{-H_>}}$$

This case is of course a bit "simple" because the Hamiltonian factors in $\vec{k}$: no couplings (interactions) between different $\vec{k}$ are present

We then integrate the > part:

$$Z = Z_> \int \mathcal{D}\phi_< \, e^{-\frac{1}{(2\pi)^d}\int_0^{\Lambda/b} (ck^2 + r)\, \tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})\, d\vec{k}}$$

Now we would like to rescale everything so that the remaining Hamiltonian looks just like the original one:

$$\int_0^{\Lambda/b} \longrightarrow \int_0^{\Lambda}$$

To do so, we change variable

$$\vec{k}' = b\vec{k} \qquad d\vec{k}' = b^d \, d\vec{k}$$

and we obtain:

$$\int_0^\Lambda d\vec{k}'\, b^{-d}\left(b^{-2}k'^2+r\right)\tilde{\phi}\left(\frac{\vec{k}'}{b}\right)\tilde{\phi}^*\left(\frac{\vec{k}'}{b}\right)$$

We assume that the field itself scales with b:

$$\tilde{\phi}\left(\frac{\vec{k}'}{b}\right)=\tilde{\phi}(\vec{k}')\, b^{\lambda}$$

Is this reasonable? We have to think that the $\tilde{\phi}$ at the new scale is a sort of coarse-grained (average) of the $\tilde{\phi}$ at the scales that we have coarse-grained. Since $\tilde{\phi}$ has zero average, its typical scale is related to the volume. If all the $\tilde{\phi}_s$ at smaller scales were uncorrelated we would expect

$$\tilde{\phi}_{c.g.}\sim\left(b^d\right)^{1/2}\tilde{\phi}_{micro}$$ from the

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If instead they were correlated, λ is to be defined.

Putting everything together:

$$\int_0^\Lambda d\vec{k}'\left(c\, b^{-2}k'^2+r\right)b^{-d}\, b^{2\lambda}\,\tilde{\phi}(\vec{k}')\,\tilde{\phi}^*(\vec{k}')$$

and this must be equal to

$$\int_0^\Lambda d\vec{k}\left(c\, k^2+r\right)\tilde{\phi}(\vec{k})\,\tilde{\phi}^*(\vec{k})$$

From this we obtain

$$\begin{cases} r' = b^{-d+2\lambda}\, r \\ c' = b^{-(d+2)+2\lambda}\, c \end{cases}$$

These are the recursion relations of the RG.

Two cases

1) $\lambda = \frac{d}{2} \Rightarrow \begin{cases} r' = r \\ c' = b^{-2} c \end{cases}$

r is automatically a fixed point!

$c' < c \quad (b > 1)$

This means that the only solution is $\boxed{c = 0}$

The fixed point is thus

$(r, 0)$

Interpretation:

$c^* = 0 \Rightarrow$ fluctuations are irrelevant (they come from $(\nabla\phi)^2$)

r finite means that the system reduces to an homogeneous field whose average value is 0

$\Rightarrow T = \infty$

This is a stable point.

2)

$$\begin{cases} r' = b^{-d+2\lambda} r \\ c' = b^{-(d+2)+2\lambda} c \end{cases}$$

now choose

$$\lambda = \frac{d+2}{2}$$

Then

$$\begin{cases} r' = b^2 r \\ c' = c \end{cases}$$

The fixed point is now $r^* = 0$ and $c > 0$

The system is completely dominated by fluctuations.

Moreover $r \sim |t|$ so that $r^* = 0 \Rightarrow |t| = 0$

critical point!

Furthermore,

$r' > r \Rightarrow$ unstable fixed point.

Last but not least, we can compute the critical exponent y_t

$$y_t = \log_b\left(\frac{dr'}{dr}\right) = \log_b b^2 = 2$$

Reminding that $y_t = \frac{1}{\nu}$ we have

$$\nu = 1/2$$ which is the result that we had previously seen.

We now know how to treat the coarse graining for the ϕ^4 model:

1) Fourier transform
2) Shell integration $(\Lambda/b, \Lambda)$
3) Rescaling of the fields
4) Choice of the rescaling factors to find the fixed points.

The ϕ^4 case is yet more complex.

Let's write the interaction, ϕ^4 term explicitely in Fourier transform: H_{int}

$$\int d\vec{x}\, \phi^4 = \frac{1}{(2\pi)^{4d}} \int d\vec{k}_1\, d\vec{k}_2\, d\vec{k}_3\, d\vec{k}_4\, \tilde{\phi}(\vec{k}_1)\tilde{\phi}(\vec{k}_2)\tilde{\phi}(\vec{k}_3)\tilde{\phi}(k_4) \underbrace{\int d\vec{x}\, e^{i(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\vec{x}}}_{2\pi\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)} =$$

$$= \frac{1}{(2\pi)^{3d}} \int d\vec{k}_1\, d\vec{k}_2\, d\vec{k}_3\, d\vec{k}_4\, \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\, \tilde{\phi}(\vec{k}_1)\tilde{\phi}(\vec{k}_2)\tilde{\phi}(\vec{k}_3)\tilde{\phi}(\vec{k}_4)$$

Including this term in the Hamiltonian,

$$Z = \int \mathcal{D}\tilde{\phi}\; e^{-H}$$

cannot be computed, but let's rewrite it as

$$Z = \int \mathcal{D}\tilde{\phi}\; e^{-\int_0^\Lambda d\vec{k}\,(ck^2+r)\tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k}) - H_{int}} =$$

$$= \int \mathcal{D}\tilde{\phi}_<\; e^{-H_<} \int \mathcal{D}\tilde{\phi}_>\; e^{-H_{2,>}}\; e^{-H_{int}}$$

where we have defined

$$H_< = \int_0^{\Lambda/b} d\vec{k}\,(ck^2+r)\tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k}) + u\int_0^{\Lambda/b} \prod_{i=1}^{4} d\vec{k}_i\, \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4) \prod_{i=1}^{4}\tilde{\phi}(\vec{k}_i)$$

and

$$H_{2,>} = \int_{\Lambda/b}^{\Lambda} d\vec{k}\,(ck^2+r)\,\tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})$$

Now we focus on

$$\int \mathcal{D}\tilde{\phi}_> \, e^{-H_{2,>}} \, e^{-H_{int_>}}$$

This contains the cases where at least one of the ks is between $\frac{\Lambda}{b}$ and Λ

and we expand $e^{-H_{int_>}}$

$$e^{-H_{int}} \simeq 1 - H_{int_>} + \frac{1}{2} H_{int_>}^2 + \dots$$

For the time being we just use $1 - H_{int_>}$

then

$$\int \mathcal{D}\tilde{\phi}_> \, e^{-H_{2,>}} (1 - H_{int_>}) = \underbrace{\int \mathcal{D}\tilde{\phi}_> \, e^{-H_{2,>}} \cdot \overbrace{\int \mathcal{D}\tilde{\phi}_> \frac{e^{-H_{2,>}}(1 - H_{int_>})}{\int \mathcal{D} \, e^{-H_{2,>}}}}^{\text{This is the average of } (1-H_{int})}}_{\text{We have multiplied and divided by } \int \mathcal{D}\tilde{\phi}_> e^{-H_{2,>}}}$$

We must compute

$$\langle 1 - H_{int_>} \rangle_{2,>}$$

average using $H_{2,>}$

$\langle 1 \rangle = 1$

$$\langle H_{int_>} \rangle_{2,>} = \langle u \int_0^\Lambda d\vec{k}_1 \, d\vec{k}_2 \, d\vec{k}_3 \, d\vec{k}_4 \; \delta(k_1+k_2+k_3+k_4)\, \tilde{\phi}(\vec{k}_1)\tilde{\phi}(\vec{k}_2)\tilde{\phi}(\vec{k}_3)\tilde{\phi}(\vec{k}_4) \rangle_{2,>}$$

There are several cases, depending on how many of the $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_3$ and $\vec{k}_4$ are between $\frac{\Lambda}{b}$ and Λ

1) None of the k_s is between $\frac{\Lambda}{b}$ and Λ ! This is the case of $H_<$

2) 1 or 3 k_s are between $\frac{\Lambda}{b}$ and Λ

In this case, since the Gaussian weigth $e^{-H_{2,<}}$ is even, the average of an odd function vanishes

3) 4 $\vec{k}_s$ are between $\frac{\Lambda}{b}$ and Λ : the average is just a constant

4) 2 $\vec{k}_s$ are between $\frac{\Lambda}{b}$ and Λ.

There are $\binom{4}{2} = 6$ such cases.

Moreover remember that the weight is $e^{-(ck^2+r)\tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})}$ which means that the two $\vec{k}_s$ between $\frac{\Lambda}{b}$ and Λ must be opposite, for example $\vec{k}_2 = -\vec{k}_1$

Then, since anyway there is $\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$ and $\vec{k}_1+\vec{k}_2=0$ we have also $\vec{k}_3+\vec{k}_4=0 \Rightarrow \vec{k}_4 = -\vec{k}_3$

Putting all together we have

$$\frac{1}{Z_{2,>}} \int \mathcal{D}\tilde{\phi}_> \, e^{-\int_{\Lambda/b}^{\Lambda} d\vec{k}\,(ck^2+r)\tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})} \cdot$$

$$\cdot \left[1 - u\int_{\Lambda/b}^{\Lambda} \prod_{i=1}^{4} d\vec{k}_i \, \tilde{\phi}(\vec{k}_1)\tilde{\phi}(\vec{k}_2)\tilde{\phi}(\vec{k}_3)\tilde{\phi}(\vec{k}_4) + \right.$$

$$\left. - 6u \int_0^{\Lambda/b} d\vec{k}\, \tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k}) \int_{\Lambda/b}^{\Lambda} d\vec{k}'\, \tilde{\phi}(\vec{k}')\tilde{\phi}^*(\vec{k}') \right] =$$

$$= 1 - u \cdot \text{const} - 6u \int_0^{\Lambda/b} d\vec{k}\, \tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k}) \cdot$$

$$\cdot \int_{\Lambda/b}^{\Lambda} d\vec{k}' \int \mathcal{D}\tilde{\phi}_> \frac{1}{Z_{2,>}} e^{-\int_{\Lambda/b}^{\Lambda}(ck^2+r)\tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})} \tilde{\phi}(\vec{k}')\tilde{\phi}^*(\vec{k}')$$

We now recall that

$$\int \mathcal{D}\tilde{\phi}_> \frac{1}{Z_{2,>}} e^{-\int_{\Lambda/b}^{\Lambda}(ck^2+r)\tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})} \tilde{\phi}(\vec{k}')\tilde{\phi}^*(\vec{k}') =$$

$$= \frac{1}{ck'^2+r}$$

Then

$$\langle 1 - H_{int,>} \rangle_{2,>} = (1 - \text{const}) \left[1 - \frac{6u}{1+\text{const}} \int_{\Lambda/b}^{\Lambda} d\vec{k}' \frac{1}{ck'^2+r} \cdot \int_0^{\Lambda/b} \tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})\, d\vec{k} \right]$$

We now write

$$\langle 1 - H_{int,>} \rangle_{2,>} = (1 - const)\, e^{\ln\left(1 - \frac{6u}{1-const}\left(\int_{\Lambda/b}^{\Lambda} d\vec{k} \frac{1}{ck^2+r}\right) \cdot \int_0^{\Lambda/b} d\vec{k}\, \tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})\right)}$$

Still in the assumption that u is very small, we have

$$\langle 1 - H_{int,>} \rangle_{2,>} = (1 - const)\, e^{-\frac{6u}{1-const}\left(\int_{\Lambda/b}^{\Lambda} d\vec{k} \frac{1}{ck^2+r}\right) \int_0^{\Lambda/b} d\vec{k}\, \tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})}$$

Now we can put it all together:

$$Z = \int \mathcal{D}\tilde{\phi}_< \, e^{-H_<} \overbrace{\int \mathcal{D}\tilde{\phi}_> \, e^{-H_{2,>}}}^{Z_{2,>} \text{ which is a constant}} \cdot (1 - const) \cdot$$

$$\cdot\, e^{-\frac{6u}{1-const}\left(\int_{\Lambda/b}^{\Lambda} d\vec{k} \frac{1}{ck^2+r}\right) \int_0^{\Lambda/b} d\vec{k}\, \tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})} =$$

$$= \underbrace{(1 - const) \cdot Z_{2,>}}_{\text{constant}} \int \mathcal{D}\tilde{\phi}_< \, e^{-\int_0^{\Lambda/b} d\vec{k}\left[ck^2 + \left(r + \frac{6u}{1-const}\int_{\Lambda/b}^{\Lambda} d\vec{k} \frac{1}{ck^2+r}\right)\right]\tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})} \cdot$$

$$\cdot\, e^{-u\int_0^{\Lambda/b} \prod_{i=1}^{4} d\vec{k}_i \prod_{i=1}^{4} \tilde{\phi}(\vec{k}_i)\, \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)}$$

Just as we did before, we have to rescale:

$$\vec{k}' = \vec{k} b \qquad \tilde{\phi}\left(\frac{\vec{k}}{b}\right) = b^{\lambda} \tilde{\phi}(\vec{k})$$

Then we have

$$Z = \text{const} \cdot \int \mathcal{D}\tilde{\phi}\, e^{-\int_0^{\Lambda} d\vec{k}\, [c' k^2 + r']\, \tilde{\phi}(\vec{k})\, \tilde{\phi}^*(\vec{k}) - u' \int_0^{\Lambda} \prod_{i=1}^{4} d\vec{k}_i \prod_{i=1}^{4} \tilde{\phi}(\vec{k}_i) \cdot \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)}$$

with

$$\begin{cases} c' = b^{-(d+2)+2\lambda} c \\ r' = b^{-d+2\lambda} \left(r + \frac{6}{1-\text{const}}\, u \int_{\Lambda/b}^{b} \frac{1}{ck^2 + r}\, d\vec{k} \right) \\ u' = \underbrace{b^{-3d+4\lambda}}\, u \end{cases}$$

3d because $\delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)$ constrains one integral

And we must find the fixed point.

Case 1

as we did previously we set $\lambda = \frac{d}{2}$

This was the case leading to $T = \infty$ as fixed point

Here it leads to

$$\begin{cases} c' = b^{-2} c \xrightarrow{b>1} 0 \\ u' = b^{-d} u \longrightarrow 0 \\ r' = r \end{cases}$$

The fixed point is then the same as before

<u>Case 2</u>

$$\lambda = \frac{d+2}{2}$$

$$\begin{cases} c' = c \\ u' = u \, b^{4-d} \\ r' = b^2 (r + \text{const} \cdot u) \end{cases}$$

This is now more interesting, because

- if $d > 4$ $\quad u' \longrightarrow 0$

  and we obtain exactly the same result as before $\Longrightarrow$ <u>mean field</u>

- if $d < 4$ then $u' \longrightarrow \infty$ and we need to go to the next order in u, because we cannot say that u is small.